Random directed polymers and their bosonic and fermionic field theories

1991 J. Phys. A: Math. Gen. 24 5321

(http://iopscience.iop.org/0305-4470/24/22/016)

J Phys A. Math. Gen 24 (1991) 5321-5325 Printed in the UK

Random directed polymers and their bosonic and fermionic field theories

T A Vilgis

Max-Planck-Institut fur Polymerforschung, Ackermannweg 10, PO Box 3148, D-6500 Mainz, Federal Republic of Germany

Received 19 December 1990, in final form 5 August 1991

Abstract. This paper derives a bosonic and fermionic field theory for randomly directed polymers The bosonic field theory needs replicas and it is shown that the problem can be mapped in an O(n) symmetric Lagrangian with an attractive ϕ^4 term The fermionic field theory introduces alternatively anticommutating fields and a supersymmetric Lagrangian is derived for the randomly directed polymer

1. Introduction

The problem of random directed polymers has been recognized as a rich problem in many branches of theoretical physics and statistical mechanics. The propagator of a polymer in a random potential is given by the following path integral (see e.g [1]).

$$G(R, N)=\int_{r(0)=0}^{r(N)=R} \mathrm{D}r(s) \exp\left\{-2l \int_0^N \left(\frac{\partial r}{\partial s}\right)^2 - \int_0^N \mathrm{d}s\, V(r(s), s)\right\} \qquad (1)$$

where R is the end point of the polymer, N its total contour length, d the space dimension, l the elementary step length, and $v(r)$ the random potential which is assumed to be of Gaussian nature, i.e.

$$\langle V(r, s)\rangle = 0 \qquad \langle V(r, s) V(r', s')\rangle = \Delta\delta(r-r')\delta(s-s'). \qquad (2)$$

It has been shown that the solution of this problem is mathematically equivalent to the growth model of the Eden type [2], a randomly stirred fluid whose hydrodynamics is approximated by Burger's equation (3). Relationships to spin glasses and travelling wavefronts have also been pointed out [4]. Moreover it has also been speculated that there exists a superuniversal law for the end to end vector, i e. $R \sim N^{2/3}$ for such problems [5a]. More recently it has been shown that this exponent reduces to $\frac{1}{2}$ as $d \to \infty$ [5b]. Polymers in random media have been revisited recently by Parisi [6] in two dimensions where it was shown that the replica symmetry is weakly broken.

The definition in (1) can be read twofold. First as it stands it defines a random directed polymer in $(d-1)$ dimensions and we have used the same definition as in [5a], i.e. the vector r refers to the $d-1$ transverse directions and the s to the longitudinal one. On the other hand it can be read as an unconstrained polymer in the dimensions in a random potential. We will, in the following, mostly deal with the second definition where we will recover several known results discussed at the end of the paper. But one can immediately generalize what we will derive with respect to the problem of the random directed polymer.

In this paper we derive field theories for the problem of such random polymers. We therefore consider a conventional approach where we use commutating fields. In this case we recover an $O(n)$ symmetric field theory with an attractive ϕ^4 term in the limit $n \to 0$. This field theory has some similarities to that of electrons in random potentials (see e.g. [7]), leading to nonlinear σ-models [8].

The second equivalent field theory uses anticommutating fields (see e.g. [9]). The advantage of that representation is that the $n \to 0$ limits can be avoided. The resulting Lagrangian including fermionic and bosonic fields is supersymmetric. Moreover it has a negative coupling constant in comparison with the fermionic Lagrangian of ordinary self-avoiding walks [10] indicating the different nature of the problem.

2. The bosonic field theory

In this section we derive the conventional field theory for randomly directed polymers. We start from (´) which reads in its normalized form

$$G(\mathbf{R}, N[V]) = \frac{\int_{\mathbf{r}(0)=\mathbf{0}}^{\mathbf{r}(N)=\mathbf{R}} \mathrm{D}\mathbf{r}(s) \exp\left\{-\frac{d}{2}\int_0^N \mathrm{d}s\left(\frac{\partial \mathbf{r}}{\partial s}\right)^2 - \int_0^N \mathrm{d}s\, V([\mathbf{r}(s)])\right\}}{\int \mathrm{D}\mathbf{r}(s) \exp\left\{-\frac{d}{2}\int_0^N \mathrm{d}s\left(\frac{\partial \mathbf{r}}{\partial s}\right)^2 - \int_0^N \mathrm{d}s\, V([\mathbf{r}(s)])\right\}} \tag{3}$$

where we have set $l=1$ and dropped the explicit s dependence of the potential for simplicity. The progpagator $G(\mathbf{R}, N, [V])$ can alternatively be generated by the differential equation, which corresponds to the path integral in the numerator [11], i.e. the diffusion equation

$$\left(\frac{\partial}{\partial N} - \frac{d}{2}\nabla^2 + V(\mathbf{r})\right) G(\mathbf{R}, N, [V]) = \delta(\mathbf{R})\delta(N) \tag{4}$$

which reads in Laplace transform with respect to N

$$\left(E - \frac{d}{2}\nabla^2 + V(\mathbf{r})\right) G(\mathbf{R}, E, [V]) = \delta(\mathbf{R}). \tag{4a}$$

Thus we can construct a field theory from the Gaussian integral

$$G(\mathbf{R}, E, [V]) = \frac{\int \delta\phi(\mathbf{r})\phi(\mathbf{R})\phi(\mathbf{0}) \exp\left\{-\int \mathrm{d}\mathbf{r} \int \mathrm{d}\mathbf{r}'\, \phi(\mathbf{r}) G^{-1}(\mathbf{r}, \mathbf{r}', E, [V])\phi(\mathbf{r})\right\}}{\int \delta\phi(\mathbf{r}) \exp\left\{-\int \mathrm{d}\mathbf{r} \int \mathrm{d}\mathbf{r}'\, \phi(\mathbf{r}) G^{-1}(\mathbf{r}, \mathbf{r}', E, [V])\phi(\mathbf{r})\right\}} \tag{5}$$

where $G^{-1}(\mathbf{r}, \mathbf{r}', E[V])$ is given by the differential equation ($4a$), and we can write for the propagator

$$\begin{aligned} G(\mathbf{R}, E, [V]) &= \frac{\int \delta\phi(\mathbf{r})\phi(\mathbf{R})\phi(\mathbf{0}) \exp\left\{-\int \mathrm{d}\mathbf{r}\, \phi(\mathbf{r})(E-\nabla^2)\phi(\mathbf{r}) - \int \mathrm{d}\mathbf{r}\, \phi^2(\mathbf{r}) V(\mathbf{r})\right\}}{\int \delta\phi(\mathbf{r}) \exp\left\{-\int \mathrm{d}\mathbf{r}\, \phi(\mathbf{r})(E-\nabla^2)\phi(\mathbf{r}) - \int \mathrm{d}\mathbf{r}\, \phi^2(\mathbf{r}) V(\mathbf{r})\right\}} \\ &= \frac{1}{Z[V]} \int \delta\phi(\mathbf{r})\phi(\mathbf{R})\phi(\mathbf{0}) \\ &\quad \times \exp\left\{-\int \mathrm{d}\mathbf{r}\, \phi(\mathbf{r})(E-\nabla^2)\phi(\mathbf{r}) - \int \mathrm{d}\mathbf{r}\, V(\mathbf{r})\phi^2(\mathbf{r})\right\}. \end{aligned} \tag{6}$$

In order to average over the Gaussian potential $V[r]$ which appears in the numerator and denominator we use the standard replica trick

$$G(\boldsymbol{R}, E, [V]) = \lim_{n\to 0} \int \delta\phi\, \phi(\boldsymbol{R})\phi(\mathbf{0}) Z^{n-1}[V] \times \exp\left\{-\int \mathrm{d}\boldsymbol{r}\, \phi(\boldsymbol{r})(E-\nabla^2)\phi(\boldsymbol{r}) - \int \mathrm{d}\boldsymbol{r}\, V(\boldsymbol{r})\phi^2(\boldsymbol{r})\right\}. \tag{7}$$

After straightforward manipulation we find the representation

$$\overline{G(\boldsymbol{R}, E)} = \lim_{n\to 0} \int \prod_{a=1}^{n} \delta\phi_a(\boldsymbol{r})\phi_1(\boldsymbol{R})\phi_1(\mathbf{0}) \exp\{-\mathscr{L}([\phi_a])\} \tag{8}$$

with the Lagrangian

$$\mathscr{L}[\phi] = \int \mathrm{d}\boldsymbol{r}\left\{\sum_{a=1}^{n} \phi_a(E-\nabla^2)\phi_a - \Delta\left(\sum_{a=1}^{n} \phi_a^2\right)^2\right\}. \tag{9}$$

This is now a ϕ^4-theory, but with an attractive interaction term $-\Delta(\phi^2)^2$. This Lagrangian is of the same type as in the field theory of the electron localization problem, except one has there two fields $\phi_\pm$ since the two particle Green function is studied (see (3.5) in [7]). The last term in the Lagrangian (9) couples two replicas of the fields and is has been shown by Parisi [6] that the replica symmetry is broken. In [6] entirely different methods have been used so that it would be interesting to derive the weak replica symmetry breaking from the field theory. It would also be interesting to connect the problem of a polymer in a random potential to the electron localization problem. There is obviously a deeper connection since the interaction term in the corresponding Lagrangians are identical. One can employ similar techniques as developed in [7] in order to obtain a nonlinear σ-model. On the other hand regularization techniques have been developed to renormalize the $g\phi^4$-theories where g is negative [12].

3. The fermionic field theory

In order to derive the fermionic field theory we start from (3)–(6) and we notice that the inverse of the partition function can be written as

$$\frac{1}{Z[V]} = \det(E - \nabla^2 + V(\boldsymbol{r})). \tag{10}$$

This determinant can be generated by an integration over anticommutating fields $\psi^+(\boldsymbol{r})$ and $\psi(\boldsymbol{r})$ and it becomes

$$\frac{1}{Z[V]} = \int \delta\psi^+\delta\psi \exp\left\{-\int \mathrm{d}^3r\, \psi^+(\boldsymbol{r})(E-\nabla^2+V(\boldsymbol{r}))\psi(\boldsymbol{r})\right\}. \tag{11}$$

Thus we can write for the propagator $G(R, E, [V])$

$$G(\boldsymbol{R}, E, [V]) = \int \delta\phi(\boldsymbol{r})\delta\psi^+(\boldsymbol{r})\delta\psi(\boldsymbol{r})\phi(\boldsymbol{R})\phi(\mathbf{0}) \times \exp\left\{-\int \mathrm{d}\boldsymbol{r}[\phi(\boldsymbol{r})(E-\nabla^2)\phi(\boldsymbol{r}) + \psi^+(\boldsymbol{r})(E-\nabla^2)\psi(\boldsymbol{r}) + V(\boldsymbol{r})(\phi^2(\boldsymbol{r}) + \psi^+(\boldsymbol{r})\psi(\boldsymbol{r}))]\right\}. \tag{12}$$

The average over the quenched random potential can now be carried out and we find

$$\overline{G(R,E)}=\int \delta\phi(r)\delta\psi^{+}(r)\delta\psi(r)\phi(R)\phi(0)\exp\left\{-\int \mathrm{d}r\,\mathscr{L}_0+\Delta\int \mathrm{d}r(\phi^2+\psi^{+}\psi)^2\right\} \quad (13)$$

which can be further treated by a Gaussian transformation, which decouples the $\phi^2\psi^+\psi$ and $(\psi^+\psi)^2$ terms to the standard form

$$\overline{G(R,E)}=\int \delta\phi(r)\delta\psi^{+}(r)\delta\psi(r)\delta\alpha(r)\exp\left\{-\int \mathrm{d}r\,\mathscr{L}_{\mathrm{susy}}(\phi,\psi^{+},\psi,\alpha)\right\} \quad (14)$$

where

$$\mathscr{L}_{\mathrm{susy}}=\phi(r)(E-\nabla^2)\phi(r)+\psi^{+}(r)(E-\nabla^2)\psi(r) + 2\sqrt{\Delta}\,\alpha(r)(\phi^2(r)+\psi^{+}(r)\psi(r))+\tfrac{1}{2}\alpha^2(r) \quad (15)$$

which is indeed of the standard supersymmetric form [10]. This Lagrangian is invariant under super-rotations. The main difference to the corresponding Lagrangian of the self-avoiding walk is the positive coupling constant Δ which is here a measure for the disorder of the potential. The supersymmetric Lagrangian for the SAW is recovered if we put $\Delta\to-\Delta$ which introduces the imaginary unit i in front of the $(\phi^2+\psi^+\psi)$ term (see [10] for the discussion).

4. Summary

In this paper we derived two field theories for randomly directed polymers. The first one was the bosonic field theory, which uses commutating fields. The denominator has been removed by the replica method before the average over the random potential has been taken. The field theory corresponds to an $O(n)$ symmetric Lagrangian $(n\to 0)$ with a negative ϕ^4, as it is observed in the electron localization problem [7]. One can decouple the $\Delta\sum_{a,b}\phi_a^2\phi_b^2$ term by introducing new (order parameter) fields Q_{ab} by a Gaussian transformation to proceed further. This can be seen as follows. We can use a Gaussian transformation

$$\exp\left(+\Delta\int \mathrm{d}^d r\sum_{a,b=1}^{n}\phi_a^2(r)\phi_b^2(r)\right) = \int\prod_{a,b=1}^{n}\mathscr{D}Q_{ab}(r) \times\exp\left(-\frac{1}{2\Delta}\int \mathrm{d}^d r\sum_{a,b=1}^{n}Q_{ab}^2(r)-\int \mathrm{d}^d r\sum_{a,b=1}^{n}Q_{ab}(r)\phi_a(r)\phi_b(r)\right) \quad (16)$$

This will lead to a field theory entirely in the field $Q_{ab}(r)$, since the ϕ-integration is Gaussian and can be carried out. The simple result is

$$G(R,E)=\lim_{n\to 0}\int\prod_{a,b=1}^{n}\mathscr{D}Q_{ab}(r)\exp\{-\mathscr{L}([Q_{ab}(r)]\} \quad (17)$$

with

$$\mathscr{L}([Q_{ab}(r)])=\sum_{a,b=1}^{n}\int \mathrm{d}^d r\left\{\frac{1}{2\Delta}Q_{ab}^2(r)+\log\mathrm{Tr}((E-\nabla^2)\delta_{ab}+Q_{ab})\right\}. \quad (18)$$

This is now of the same form as the field theory derived in [7]. The consequences of this will be left to an extended paper.

The second type of field theory introduced two fermionic fields ψ^+, ψ to remove the determinant in the denominator of the Green function. We derived a simple supersymmetric Lagrangian analogously to that of the ordinary self-avoiding walk [10] but with an effective attractive coupling constant, which signals the different nature of the problem. The random potential produces an attractive interaction. This can be seen directly if we write the partition function of the polymer in the random potential

$$Z=\int \delta r(s)\exp\left\{-\frac{d}{2}\int_0^N\left(\frac{\partial r}{\partial s}\right)^2-\int_0^N V(r(s))\right\} \tag{19}$$

and calculate the quenched average by the replica method, i.e. $\overline{\log Z}=\lim_{n\to 0}(\overline{Z^n}-1)/n$. We find immediately:

$$\overline{Z^n}=\int\prod_{a=1}^{n}\delta r_a(s)\times\exp\left\{-\sum_{a=1}^{n}\int_0^N\left(\frac{\partial r_a}{\partial s}\right)^2\mathrm{d}s+\Delta\sum_{a,b}\int_0^N\int_0^N\delta(r_a(s)-r_b(s'))\,\mathrm{d}s\,\mathrm{d}s'\right\} \tag{20}$$

i.e. n random walks which attract each other and themselves with a short range potential of strength Δ. This attraction is also present in the corresponding field theories which are derived in this paper. In (19) we have the corresponding theory in $R(s)$ space, which has been studied by several authors. For example the same theory as given in (13) has been obtained by the study of the problem of polymers at the presence of fixed obstacles [13]. The authors in [13] used a variational principle to approximate the path integral given in equation (19). The corresponding annealed problem (moving obstacles) has been considered in [14]. The problem has also been addressed by scaling arguments of the Flory type (see e.g. [15, 16] and references therein).

The next step is now to use the field theories derived in this paper to calculate physical observables. This will be done in an extended paper.

References

[1] Kardar M and Zhang Y-C 1987 *Phys Rev Lett* **58** 2087
[2] Kardar M, Parisi G and Zhang Y-C 1986 *Phys. Rev Lett* **56** 889
[3] Forster D, Nelson D R and Stephen M J 1977 *Phys Rev Lett* A **16** 732
[4] Derrida B and Spohn H 1988 *J Stat. Phys* **51** 817
[5a] McKane A J and Moore M A 1988 *Phys Rev. Lett.* **60** 527
[5b] Kim J M and Kosterlitz J M 1989 *Phys Rev Lett* **62** 2289
[6] Parisi G 1990 *J Physique* **51** 1595
[7] McKane A J and Stone M 1981 *Ann Phys* **131** 36
[8] Wegner F J 1982 *Anderson Localisation* ed Y Nagooka and H Fukuyama (Berlin. Springer)
[9] Itzykon C and Drouffe J M 1989 *Statistical Field Theory* vols 1, 2 (Cambridge. Cambridge University Press)
[10] Parisi G and Sourlas N 1980 *J. Physique Lett* **41** L403
[11] Doi M, Edwards S F 1986 *The Theory of Polymer Dynamics* (Oxford. Oxford University Press)
[12] McKane A J, Wallace D J and de Alcantara Bonfim O F 1984 *J. Phys A: Math. Gen.* **17** 1861
[13] Edwards S F and Muthukumar M 1988 *J Chem. Phys* **89** 2435
[14] Thirumalai D 1988 *Phys Rev* A **37** 269
[15] Machta J and Guyer R A 1989 *J Phys A Math Gen* **22** 2539
[16] Halpin-Healey T 1989 *Phys Rev Lett* **62** 442